The Root Locus Method

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Outline

- The Root Locus Method
- Closed-loop poles
- Plotting the root locus of a transfer function
- Choosing a value of K from root locus
- Closed-loop response
- Key MATLAB commands used: feedback, rlocfind, rlocus, sgrid, step, pzmap, zpk, rltool
Root Locus Method

- Closed-loop response depends on the location of closed-loop poles.

- If system has a variable design parameter (e.g., a simple gain adjustment or the location of compensation zero), then the closed-loop pole locations depend on the value of the design parameter.

- The root locus of a system is the plot of the paths (loci) of all possible closed loop poles as the design parameter takes on a range of possible values.
The poles that provide the desired closed-loop response are selected and the proper value of the design parameter is thereby established.

The closed-loop poles are the roots of the system's characteristic equation. Since finding the roots of polynomials of degree higher than 3 is laborious, graphical aids were devised in the late 1940s to help construct the root loci.

Recently, computer-aided design tools such as Matlab provide a convenient computer solution.

The older, graphical aids are still relevant since the ability to quickly sketch root loci by hand is invaluable in making fundamental decisions early in the design process and in checking Matlab results.
Closed Loop Poles

- Closed-Loop Transfer Function

\[ \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \]

- Where \( G(s) \) is the forward-path transfer function and \( H(s) \) is the feedback-path transfer function

- Characteristic Equation

The poles of the closed loop system are values of \( s \) such that

1. \( 1 + G(s)H(s) = 0 \) or
2. \( G(s)H(s) = -1 \)

- If \( G(s)H(s) = \frac{k\ num(s)}{\ den(s)} \), then this equation has the form:

\[ \text{den}(s) + k \ \text{num}(s) = 0 \]

\[ \frac{\text{den}(s)}{k} + \text{num}(s) = 0 \]
Geometric Properties

\[ 1 + K \frac{\text{Num}(s)}{\text{Den}(s)} = 0 \]
\[ K \frac{\text{Num}(s)}{\text{Den}(s)} = -1 \]

- **Complex number => Amplitude and angle**

  \[ |K \frac{\text{Num}(s)}{\text{Den}(s)}| = 1 \]

  \[ \angle K \frac{\text{Num}(s)}{\text{Den}(s)} = 180^\circ \pm 360^\circ \]

  - **Magnitude Condition or Amplitude relation**
  - **Angle Condition or Angle relation**
Example

\[ G(s)H(s) = \frac{K_i \prod_{k=1}^{m} (\tau_k s + 1)}{s^n \prod_{j=N+1}^{n} (\tau_j s + 1)} \]

\[ \left| \frac{K_i \prod_{k=1}^{m} (\tau_k s + 1)}{s^n \prod_{j=N+1}^{n} (\tau_j s + 1)} \right| = 1 \ \& \]

\[ \sum_k \angle(\tau_k s + 1) - N \angle s - \sum_j \angle(\tau_j s + 1) = \begin{cases} 
(1 + 2q) \ast 180^\circ & \text{for } K_i > 0 \\
2 \ast q \ast 180^\circ & \text{for } K_i < 0
\end{cases} \]
Questions 1

- As $0 \leq K \leq \infty$, Where the root locus starts and where it ends.

\[ EX1: \quad 1 + K \frac{1}{s(s + 2)} \]

\[ EX2: \quad 1 + K \frac{s + 3}{s + 1} \]

\[ EX3: \quad 1 + K \frac{(s + 1)(s + 2)}{s(s + 3)} \]
As $K$ ranges from 0 to infinity the closed-loop poles migrate from the open-loop poles to the open-loop zeros. The path of a closed-loop pole on the $s$-plane is called a \textit{branch} of the root locus.

No matter what we pick $k$ to be, \textbf{the closed-loop system must always have $n$ poles}, where $n$ is the number of poles of $GH(s)$. The root locus must have $n$ branches, each branch starts at a pole of $GH(s)$ and goes to a zero of $GH(s)$.

If $GH(s)$ has more poles than zeros (as is often the case), $m < n$ and we say that $GH(s)$ has \textbf{zeros at infinity}. In this case, the limit of $GH(s)$ as $s \to \infty$ is zero. \textbf{The number of zeros at infinity is $n-m$}, the number of poles minus the number of zeros, and is the number of branches of the root locus that go to infinity (asymptotes).
Question 2

As $0 \leq K \leq \infty$, What is the part of the real axis that belongs to the Root locus.

$$EX1: \quad 1 + K \frac{(s + 1)(s + 2)}{s(s + 3)} = 0$$
The closed-loop transfer function is

\[
C(s) = \frac{K}{R(s)} = \frac{K}{s(s + 2) + K}
\]

The characteristic equation is

\[
s^2 + 2s + K = 0
\]

Consider the characteristic roots as \( K = 0 \rightarrow \infty \).
Root Locus Example

\[ s = -1 \pm \sqrt{1 - K} \]

- For \( K = 0 \) the closed-loop poles are at the open-loop poles.

- For \( 0 < K < 1 \) the closed-loop poles are on the real axis.

- For \( K > 1 \) the closed-loop poles are complex, with a real value of \(-1\) and an imaginary value increasing with gain \( K \).
Root Locus Example: Step Responses

Step Responses

1. $K = 50$.

2. $K = 2.0$.

3. $K = 15.0$.

4. $K = 1.0$.

5. $K = 0.5$.

6. $K \to \infty$.

Amplitude vs. Time (sec.)
Idea
Answer for Q2

- **Root Loci on the Real Axis:** For \( k \geq 0 \) (\( k \leq 0 \)) branches on the real axis lie to the left (right) of an odd number of open-loop poles and zeros.

- **Example 1:** Consider the open-loop transfer function,

\[
G(s)H(s) = \frac{K(s+7)}{s(s+5)(s+15)(s+20)} \quad \text{with} \quad K \geq 0
\]
Symmetry about Real Axis

- **Symmetry**: Since roots of the characteristic equation occur in complex conjugate pairs, the root loci are symmetrical with respect to the real axis.
Question 3: Asymptotic Behavior

- The loci migrate to infinity \((m<n)\) along asymptotes with the following characteristics:

  \[
  \text{Real	extunderscore Axis	extunderscore Intercept} = \sum_{j=1}^{n} -p_j - \sum_{i=1}^{m} -z_i \\
  n-m
  \]

  where \(\{-p_j\}\) and \(\{-z_i\}\) are open-loop pole and zeros

- \(q = 0, +/-1, +/-2,\ldots\)

  \[
  \text{Angle	extunderscore of	extunderscore the	extunderscore Asymptotes} = \begin{cases} 
  (2q + 1)180^\circ / (n - m) & \text{for } K > 0 \\
  2q180^\circ / (n - m) & \text{for } K < 0
  \end{cases}
  \]
Question 4: Crossing of the Imaginary axis

The value of $K$ which yields imaginary closed-loop poles can be found from the last three rows of the R-H array of the characteristic equation, i.e.,

$$
\begin{array}{ccc}
  s^2 & e_1(K) & e_2(K) \\
  s^1 & f(K) \\
  s^0 & g(K)
\end{array}
$$

The value of $K$ such that $f(K) = 0$ is that which establishes imaginary closed-loop poles.

Forming the auxiliary equation:

$$e_1(K)s^2 + e_2(K) = 0$$

Solving for $s$ yields points where the root loci cross the imaginary axis.
Angle of departure

7. The angles of departure, $\theta_d$ from poles and arrival, $\theta_a$ to zeroes may be found by applying the angle condition to a point very near the pole or zero.

- The angle of arrival at the zero, $-z_1$ is obtained from

$$\theta_{az1} + \sum_{i=2}^{m} \angle(-z_1 + z_i) - \sum_{i=1}^{n} \angle(-z_1 + p_i) = (2k + 1)\pi$$
Question 5: leaving or entering the real axis Break away Point

- Real axis breakaway points occur at maxima and minima of $K(s)$.

- Solutions of the equation $\frac{dK(s)}{ds} = 0$ yield the breakaway points

- $K(s) = - \frac{\text{den}(\text{GH}(s))}{\text{num}(\text{GH}(s))}$
Breakaway Points:

- Breakaway Points:
  When two or more loci meet, they will breakaway from this point at particular angles. The point is known as a breakaway point. It corresponds to multiple roots.
Breakaway Points:

9. The angle of breakaway is $180^\circ/k$ where $k$ is the number of converging loci.
The location of the breakaway point is found from:

$$\frac{dK}{ds} = 0 \quad \text{or} \quad \frac{d[GH(s)]}{ds} = 0$$

- **Note:**
  \[ K = -[GH(s)]^{-1} \]

\[ \frac{dK}{ds} = [GH(s)]^2 \frac{d[GH(s)]}{ds} = 0 \]

- **Also,**

\[ \frac{d[GH(s)]}{ds} = \frac{d[N(s)/D(s)]}{ds} \]
\[ = \frac{N'(s)}{D(s)} - \frac{N(s)D'(s)}{D(s)^2} = 0 \]
\[ D(s)N'(s) - N(s)D'(s) = 0 \]
Root Locus Plot: Breakaway Point Example

Consider the following loop transfer function.

\[ GH(s) = \frac{K}{s(s + 3)^2} \]

- **Real axis loci** exist for the full negative axis.
- **Asymptotes:**
  - angles = \((2k+1)\pi = \pi/3 , \pi, 5\pi/3\)
  - \[ \sigma_a = \frac{(-3-3-0)-(0)}{3} = -2 \]
Determine the breakaway points from

\[
\frac{d}{ds}\left[\frac{K}{s(s + 3)^2}\right] = \frac{d}{ds}\left[\frac{K}{s^3 + 6s^2 + 9s}\right]
\]

\[
= \frac{-K(3s^2 + 12s + 9)}{(s^3 + 6s^2 + 9s)^2} = 0
\]

then

\[
s^2 + 4s + 3 = (s + 1)(s + 3) = 0
\]

\[
s = -1, -3
\]
Question 6: Angle of departure and angle of arrival

- Departure angle from $p_2$.
  - $\theta_z = \tan^{-1}(2/3) = 33.7^\circ$
  - $\theta_{p1} = \tan^{-1}(2/-1) = 116.6^\circ$
  - $\theta_{p3} = 90^\circ$

- Then
  - $33.7^\circ - (90^\circ + 116.6^\circ + \theta_{p2}) = 180^\circ$
  - $\theta_{p2} = -352.9^\circ = +7.1^\circ$
How Can we determine the required $K$ for a particular pole?

10. For a point on the root locus, $s = s_1$

calculate the gain, $K$

from

$$ |K| = \frac{|s_1 + p_1| |s_1 + p_2|}{|s_1 + z_1| |s_1 + z_2|} \cdots $$

Alternately, $K$ may be determined graphically from the root locus plot.

$$ |K| = \frac{BCD}{A} $$